

Vector fields on C^* -algebras, semigroups of endomorphisms and gauge groups

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Abstract

What is a vector field on a C^* -algebra is defined. Its relation to semigroups of $*$ -endomorphisms was researched. Some results given about those vector fields and semigroups. There are also various constructions of semigroups including one parametrized by the cone of the future in $(3+1)$ -dimensional Minkowski space. Physical interpretations will be presented in a separate paper.

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0.1 Introduction

The formalism of C^* -algebras is known to be used for quantization and quantum observables for a long time. Several ideas of differential geometry, such as bundles, differential forms and integrals, admit their translations to algebraic language. This paper presents one concept which was apparently missing: vector fields. In Section 1 we define real vector fields on a C^* -algebra, a continuous semigroup of $*$ -endomorphisms of the same algebra, the relation between them, and basic examples. In Section 2 we present some less trivial constructs, such as the commutator of vector fields and a semigroup parametrized by several real variables. Section 3 gives the construction of vector fields via representations of the algebra. Section 4 gives some implications to semigroups, if there exist ones generated by vector fields from Section 3.

0.2 Basic facts about C^* -algebras

C^* -algebra \mathcal{A} is a complex Banach algebra with an antilinear involution, which reverses the order of multiplication: $(ab)^* = b^*a^*$. The norm must satisfy:

$$\|a\|^2 = \|a^*a\|$$

\mathcal{A}_{herm} is a real linear subspace of self-conjugate (Hermitian) elements.

A $*$ -endomorphism does not have to keep the norm. Though, one can prove that a $*$ -endomorphism cannot increase the norm of an element:

$$\|E(a^* a)\| = \|E(a)\|^2, \quad \|E(a^* a a^* a)\| = \|E(a)\|^4, \quad \|E((a^* a)^4)\| = \|E(a)\|^8, \quad \dots$$

1 Vector fields and real-parametrized endomorphism semigroups

1.1. A real-parametrized semigroup (or group) of $*$ -endomorphisms is a family of $*$ -endomorphisms parametrized by non-negative (or real) numbers, such that: $E_\zeta \circ E_\eta = E_{\zeta+\eta}$ and $E_0 = \text{Id}$. These semigroups will be referred simply as “endomorphism semigroups”. Possible example is $E_t(q) = \exp(-itg)q \exp(itg)$, $g \in \mathcal{A}_{\text{herm}}$, and it is actually the only possible example in finite dimensions. If the algebra is infinite-dimensional, then other semigroups may exist, and for a semigroup not defined by a conjugation with an exponent we cannot compute the derivative $\frac{d}{dt}E_t(q)$ for any element q . For most constructions of a semigroup (it’s not a theorem), though, we can find such $*$ -subalgebra dense in \mathcal{A} that $*$ -endomorphisms of a semigroup preserve it and aforementioned derivative is defined for any q from the subalgebra. It is obvious that such semigroup, restricted to this subalgebra, satisfies the following ordinary differential problem:

$$\forall q \in \mathcal{A}_V: \frac{d}{dt}E_t(q) = V(E_t(q)), \quad \forall t: E_t: \mathcal{A}_V \rightarrow \mathcal{A}_V \text{ a bounded linear map, } E_0 = \text{Id}, \quad (!)$$

where \mathcal{A}_V is a $*$ -subalgebra dense in \mathcal{A} and $V: \mathcal{A}_V \rightarrow \mathcal{A}$ is a fixed (unbounded) linear operator. Another way to express this is a differential equation $\dot{E} = V \circ E$ where the differentiation is understood in the strong operator topology (not norm topology) on bounded linear operators on \mathcal{A}_V . Are there necessary conditions on V ? First, it must commute with $*$, because any $*$ -endomorphism has to preserve this operation. Second, by differentiation of the condition (!) we obtain the product rule:

$$\forall u, v \in \mathcal{A}_V: V(uv) = uV(v) + V(u)v \quad (')$$

In other words, the operator V must be a differentiation on \mathcal{A}_V .

1.2. Definition: Let \mathcal{A}_V be a dense $*$ -subalgebra, and $V: \mathcal{A}_V \rightarrow \mathcal{A}$ is a linear operator satisfying the equation ('). Then V is referred to as a *vector field* on \mathcal{A} . If more precision is required, the one can say that V is a vector field on \mathcal{A} defined in the subalgebra \mathcal{A}_V . A vector field which commutes with $*$ will be referred to as a *real* vector field.

1.3. Theorem: Given a real vector field V defined in \mathcal{A}_V , an $\varepsilon > 0$ and a solution of the equation (!) for $t \in [0, \varepsilon]$. Then:

- The solution is a family of $*$ -endomorphisms of \mathcal{A}_V (and, hence, of \mathcal{A} by continuity).
- The solution, in given conditions, is unique.

- The solution extends to any $t \in [0, +\infty)$.
- The solution is continuous by both q (on all \mathcal{A}) and t . In other words, E^V_t is continuous in the (strong) operator topology.

Note: The restriction that solutions are looked for in an operator space, not as individual trajectories, allows to bypass uniqueness problems of ordinary differential equations.

Proof. From the property (*) of V follows a differential equality for a solution E of (!):

$$\forall u, v \in \mathcal{A}_V : \frac{d}{dt} E_t(uv) = \frac{d}{dt} (E_t(u) E_t(v))$$

By integration of it by t from $t = 0$ we get that $E_t(uv)$ always equals to $E_t(u) E_t(v)$ in \mathcal{A}_V , which in turn extends to all \mathcal{A} by continuity. Conservation of $*$ is proved similarly from the condition that V is real. Suppose two solutions E and \tilde{E} . Then, for any $\kappa \in \mathbb{C}$: $(1 - \kappa)E + \kappa\tilde{E}$ is also a solution of (!), each of which we already know to be a family of endomorphisms. For given t the dependence on κ can either define a complex line or a fixed endomorphism (if there is no dependence). But complex lines of $*$ -endomorphisms are impossible: either because they preserve $*$, or because each of them has the norm not greater than 1. Hence, for any t : $E_t = \tilde{E}_t$.

Extension to all $[0, +\infty)$ is made by composition, from the Archimedean property of real numbers and uniqueness of the solution. Continuity by t for any $q \in \mathcal{A}$ can be proved from density of \mathcal{A}_V and $|E_t(q)| \leq |q|$ inequality, for example, by direct verification of ε - δ definition of continuity. \square

Hence, for any real vector field V there are only two possibilities: either it defines a semigroup of endomorphisms of \mathcal{A} , or the equation (!) has no solution anywhere for $t > 0$. The former case will be referred to as “ V generates a semigroup”, or “ V is semigroup-generating”, and this semigroup will be denoted as E^V . Informally, E^V_t is the exponential of tV .

The only obvious property of the set of semigroup-generating real vector fields defined in \mathcal{A}_V is that this set is a cone.

1.4. Proposition: If a vector field V generates a semigroup, then it commutes with all its endomorphisms:

$$\forall t \geq 0 : V \circ E^V_t = E^V_t \circ V.$$

Proof: First of all, sides of this equality belong to the same space of operators from \mathcal{A}_V to \mathcal{A} , because E^V preserves \mathcal{A}_V . From (!):

$$V \circ E^V_t = \frac{d}{d\tau} E^V_{\tau} |_{\tau=t} = E^V_t \circ \left(\frac{d}{d\tau} E^V_{\tau} |_{\tau=0} \right) = E^V_t \circ V$$

where the middle equality follows from the fact that E^V is a real-parametrized semigroup of (linear) endomorphisms. The derivative is understood as pointwise on elements of the $*$ -subalgebra \mathcal{A}_V . Or, more “scientifically”, in the strong operator topology from \mathcal{A}_V to \mathcal{A} . \square

1.5. Example: let M be a real \mathcal{C}^1 -smooth manifold. We'll denote as $\mathcal{C}_0(M)$ the algebra of continuous complex-valued functions vanishing at infinity with pointwise operations multiplication and $*$ (as the complex conjugation). "Vanishing at infinity" means that $|f(\cdot)| \geq \varepsilon$ must be compact for any $\varepsilon > 0$ – this property is important to get a C^* -algebra on a manifold which is not compact. Then any vector field V_M on M translates to the vector field on $\mathcal{C}_0(M)$ defined in the $*$ -subalgebra $\mathcal{C}_0^1(M)$ of continuously differentiable functions (vanishing at infinity with the derivative, if needed) by tautological pointwise scalar product

$$V_{\mathcal{C}^1}(f)(\xi) := (df)(\xi) V_M(\xi)$$

of the 1-form of derivative of f and the given vector field on M . If the differential equation $\dot{\xi} = V(\xi)$ has for $t > 0$ the family of solutions depending continuously on the initial condition $x(0)$, namely, $F : M \times [0, +\infty) \rightarrow M$ such that

$$\forall \xi_0, t : \frac{\partial}{\partial t} F(\xi_0, t) = V(F(\xi_0, t)); \quad F(\xi_0, 0) = \xi_0,$$

then the corresponding endomorphism semigroup of $\mathcal{C}_0(M)$ is expressed as:

$$(E_t^V(f))(\xi) = f(F(\xi, t))$$

This admits generalizations beyond \mathcal{C}^1 -smoothness (say, to functions with cusps in a point where V is not defined) with appropriate tuning of the $*$ -subalgebra.

1.6. Theorem: for the commutative C^* -algebra $C_0(\mathbb{R})$ of continuous functions vanishing at infinity and its $*$ -subalgebra $C_0^1(\mathbb{R})$ of continuously differentiable functions, the set of semigroup-generating real vector fields defined in $C_0^1(\mathbb{R})$ is not convex.

Proof:

$$V_1 = (\sqrt[3]{x} + 1) \frac{d}{dx}, \quad V_2 = (\sqrt[3]{x} - 1) \frac{d}{dx}$$

Both vector fields generate semigroups, and even groups.

For $x > -1$ it may be expressed with a C^1 -substitution:

$$y := \int \frac{1}{V_1} = \int \frac{dx}{\sqrt[3]{x} + 1} = 3(\ln(\sqrt[3]{x} + 1) - \sqrt[3]{x} + \frac{1}{2}x^{\frac{2}{3}}), \quad y \in \mathbb{R}$$

which gives $V_1 = d/dy$ and $(E^{V_1}_t(f))(y) = f(y + t)$. Note that the C^1 -smoothness of y relatively to x in $x = y = 0$ is not obvious in the given expression, but can be verified by differentiation by x . The similar expression (infinitely smooth this time) exists for the action of E^{V_1} on $x < -1$. V_2 gives same substitutions save for replacement of x with $-x$ and $x > -1$ with $x < 1$. Indeed, the field

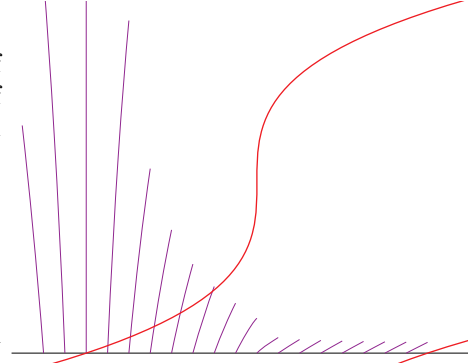
$$\frac{V_1 + V_2}{2} = \sqrt[3]{x} \frac{d}{dx}$$

is not a semigroup-generating, since the differential equation $\dot{x} = \sqrt[3]{x}$ with the initial condition $x(0) = 0$ has many solutions at $t > 0$, for example $x = \pm(\frac{2}{3}t)^{\frac{3}{2}}$, where the solution with the “+” sign is continuous with solutions for $x(0) > 0$ and one with the “-” sign is continuous with solutions for $x(0) < 0$. This obliges $(E^{(V_1+V_2)/2}_t(f))(0)$ to take values of f simultaneously from positive and negative points, hence such endomorphism is not possible. \square

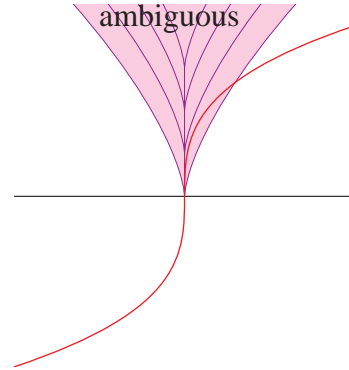
Certainly, the same is true for the algebra C_0 of any real interval (of whatever type: closed or open).

Replacement of C^1 with C^k or C^∞ does not change much.

The set of semigroup-generating vector fields is not convex even in the simplest case of infinite-dimensional C^* -algebra and the most natural choice of a dense $*$ -subalgebra: smooth functions. Henceforth, one should not expect that



Red: graphs $y = V(x)dx$ for V_1, V_2 .
Purple: integral curves for V_1 .



$y = V(x)dx$ for the red curve.
 $y = t$ for purple integral curves.

this set would be convex in more complicated algebras. This theorem means that if we are, for some reason, interested in semigroup, then it cannot be constructed easily from generic real vector fields, even if they generate semigroups.

2 Endomorphism semigroups and gauge groups

2.1. Note that any element g of the algebra defines the vector field, defined in all \mathcal{A} , by adjoint representation as a Lie algebra: $\text{ad}_g(\cdot) := [g, \cdot]$. Somewhat counter-intuitively, skew-Hermitian ($g + g^* = 0$, i.e. *imaginary*) elements define *real* vector fields,¹ and, vice versa, Hermitian elements define “imaginary” (i.e. exchanging Hermitian and skew-Hermitian elements) vector fields, because the commutator of two Hermitian elements is skew-Hermitian.

Any real vector field on \mathcal{A} defined by ad_g (i.e. g must be skew-Hermitian: $g + g^* = 0$) generates a group of $*$ -endomorphisms. If the algebra has “1”, then this group can be obtained via exponentiation. A C^* -algebra is a Banach algebra, and we can compute the exponent of an element, which of skew-Hermitian element will be unitary: $(\exp(g))^* \exp(g) = 1 = \exp(g) (\exp(g))^*$. Unitary elements define (internal) automorphisms of the algebra by conjugation;² see below.

2.2. From here onwards we’ll assume that \mathcal{A} is a unital algebra. Let \mathcal{G} be a closed subset of unitary elements, i.e. such u that $u^* u = 1 = u u^*$, which forms a group under multiplication. Then $\text{Ad}_u(\cdot) := u \cdot \cdot u^*$ is its action on \mathcal{A} by C^* -automorphisms. Denote as \mathfrak{g} the tangent space of \mathcal{G} in 1. It is easy to check that it is a closed linear subspace,³ that $\mathfrak{g}^{\mathbb{C}}$ is closed under the commutator operation, but it is not, generally, a $*$ -subalgebra.

Though, exponents of elements of $\mathfrak{g}^{\mathbb{R}}$ belong to \mathcal{G} , because it is topologically closed.

2.3. Definition: Let E be an endomorphism semigroup, and \mathcal{G} be a closed connected group of unitary elements. If E_t preserves the tangent space \mathfrak{g} ⁴ for any $t > 0$, then we’ll refer to \mathcal{G} as a *gauge group* for E .

Note that a $*$ -endomorphism (if it is not an isomorphism) usually does not map \mathcal{G} to \mathcal{G} , even if it preserves its tangent space \mathfrak{g} .

Definition: Let V be a real vector field generating an endomorphism semigroup E^V , \mathcal{G} be its gauge group, and let the intersection $\mathfrak{g} \cap \mathcal{A}_V$ be dense in \mathfrak{g} . Then we’ll refer to \mathcal{G} as a gauge group for the vector field V .

In this case, for any $g \in \mathfrak{g} \cap \mathcal{A}_V$ the derivative $\frac{d}{dt} E_t^V(g)|_{t=0}$ (which equals to $V(g)$ by 1.3.) belongs to \mathfrak{g} , which means that $\forall g \in \mathfrak{g} \cap \mathcal{A}_V : V(g) \in \mathfrak{g}$. The

¹ Apparently, the source of well-knowns disparateness in real/imaginary factors between mathematicians (who follow algebraic intuition) and physicists (who might follow differential-geometric intuition).

² If there is no “1” in \mathcal{A} , then such automorphisms also exist, although this is not a conjugation in such case.

³ Over real numbers \mathfrak{g} is a closed subspace of skew-Hermitian elements ($g + g^* = 0$). Of course, the complex tangent space is the complexification of the real tangent space.

⁴ It is not important whether E has to preserve real or complex tangent space \mathfrak{g} , because any skew-Hermitian element always maps to a skew-Hermitian one by a $*$ -endomorphism.

author currently does not know whether this condition on V is sufficient to have a gauge group \mathcal{G} .

2.4. Theorem: Let V be a semigroup-generating real vector field with a gauge group \mathcal{G} . Then:

- The family $\text{Ad}_u \circ E^V_t$ of *-endomorphisms, where $u \in \mathcal{G}$, $t \geq 0$, is a semigroup.
- For every $g \in \mathfrak{g}$ the sum of vector fields $V + \text{ad}_g$ generates an endomorphism semigroup which lies within aforementioned two-parameters semigroup.

Proof: Due to associativity of the composition, to prove the first statement it's sufficient to demonstrate that

$$\forall t \geq 0, \forall \tilde{u} \in \mathcal{G} : \exists u \in \mathcal{G} : \text{Ad}_u \circ E^V_t = E^V_t \circ \text{Ad}_{\tilde{u}}$$

Since \mathcal{G} is connected, we can check it only in a neighborhood of 1, say, for $\|\tilde{u} - 1\| < 1$ where natural logarithm is defined by its Taylor series:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

After a trivial observation that logarithm of \tilde{u} has to belong to \mathfrak{g} it becomes obvious that $u = \exp(E^V_t(\ln \tilde{u}))$ is the solution – it directly follows from the fact that E^V_t is a *-endomorphism. We also got that the dependence of u on both \tilde{u} and t is continuous.

Now we can realize that the semigroup generated by $V + \text{ad}_g$ is expressed as

$$E^{V + \text{ad}_g}_t = \text{Ad}_{\exp(\int_0^t E^V_{t-\tau}(g) d\tau)_+} \circ E^V_t \quad (\sim)$$

where $\exp(\dots)_+$ is the ordered exponential.[3] For $0 \leq s \leq t$, it satisfies the identity:

$$\exp(\int_0^t E^V_{t-\tau}(g) d\tau)_+ = \exp(\int_0^s E^V_{s-\tau}(g) d\tau)_+ \exp(\int_0^{t-s} E^V_{t-s-\tau}(g) d\tau)_+$$

On the other hand, since E^V_s is an endomorphism of the algebra, we have:

$$\begin{aligned} \text{Ad}_{\exp(\int_0^{t-s} E^V_{t-s-\tau}(g) d\tau)_+} &= \text{Ad}_{\exp(E^V_s \int_0^{t-s} E^V_{t-s-\tau}(g) d\tau)_+} = \\ &= E^V_s \circ \text{Ad}_{\exp(\int_0^{t-s} E^V_{t-s-\tau}(g) d\tau)_+} \end{aligned}$$

With it, we immediately see that compositions of *-endomorphisms from (\sim) behave as they must do in a semigroup. So, ordering in the “abnormal” direction,

higher semigroup-parameter factors to right and lower to left, is necessary to put factors with semigroup parameter close to t adjacent to the E^V_t endomorphism.

Initial condition from (!) is evident. The only thing we need to complete the proof of (\sim) is to compute its derivative. Since we already know that the formula (\sim) defines a semigroup, we can do it only at $t = 0$:

$$\begin{aligned} & \frac{d}{dt} \left(\text{Ad}_{\exp(\int_0^t E^V_{t-\tau}(g) d\tau)} \circ E^V_t \right)_{t=0} = \\ & = \text{ad} \frac{d}{dt} \exp(\int_0^t E^V_{t-\tau}(g) d\tau)_{t=0} + V = \text{ad } g + V . \end{aligned}$$

□

2.5. Commutator of vector fields cannot be defined straightly since we defined the codomain of V to be a broader space than its domain. Though, if there are two fields $Y : \mathcal{A}_Y \rightarrow \mathcal{A}$ and $Z : \mathcal{A}_Z \rightarrow \mathcal{A}$ and there exists such dense $*$ -subalgebra \mathcal{A}_{YZ} that is a subset of both \mathcal{A}_Y and \mathcal{A}_Z , Y maps it into \mathcal{A}_Z and Z maps it into \mathcal{A}_Y , then the commutator may be defined in this narrower $*$ -subalgebra:

$$[Y, Z] := Y \circ Z - Z \circ Y$$

The (\cdot) condition (see 1.1.) for $[Y, Z]$ is satisfied:

$$\begin{aligned} [Y, Z](uv) &= Y(uZ(v) + Z(u)v) - Z(uY(v) + Y(u)v) = \\ &= uY(Z(v)) + Y(u)Z(v) + Z(u)Y(v) + Y(Z(u))v - uZ(Y(v)) - Z(u)Y(v) - Y(u)Z(v) - Z(Y(u))v = \\ &= u[Y, Z](v) + [Y, Z](u)v \end{aligned}$$

Since real vector fields commute with $*$, it is obvious that the commutator of two real vector fields is *real*; see notes in 2.1.

2.6. Lemma: Let V_1 be a semigroup-generating real vector field defined in a dense $*$ -subalgebra \mathcal{A}_V . Let V_2 be a vector field such that the commutator $[V_1, V_2]$ is defined in some (narrower) dense $*$ -subalgebra $\mathcal{A}_{V_1 V_2}$ and is a bounded operator. Then:

$$[E^{V_1}_t, V_2] = \int_0^t E^{V_1}_{t-\tau} \circ [V_1, V_2] \circ E^{V_1}_\tau d\tau$$

Note that if $[V_1, V_2] = \text{ad}_c$ for certain $c \in \mathcal{A}$, then

$$\int_0^t E^{V_1}_{t-\tau} \circ [V_1, V_2] \circ E^{V_1}_\tau d\tau = \text{ad}_{\int_0^t E^{V_1}_\tau(c) d\tau} \circ E^{V_1}_t$$

because E^{V_1} consists of endomorphisms of the algebra.

Proof: First we have to ensure that the operator family $[E^{V_1}_t, V_2]$ is continuous by t in the strong operator topology from $\mathcal{A}_{V_1 V_2}$ to \mathcal{A} . It is obvious for the $E^{V_1}_t \circ V_2$ term, but for $V_2 \circ E^{V_1}_t$ we have to check that $E^{V_1}_t$ preserves $\mathcal{A}_{V_1 V_2}$, which follows from the boundness of $[V_1, V_2]$.⁵ Then, the formula can be checked by interpolation between $V_2 \circ E^{V_1}_t$ and $E^{V_1}_t \circ V_2$ by $E^{V_1}_\tau \circ V_2 \circ E^{V_1}_{t-\tau}$ (say, for $\tau = kt/n$, $k = 1 \dots n-1$, like in Riemann integral), using continuity of $E^{V_1}_t$ by t . \square

2.7. Theorem: In all conditions of 2.6., let V_2 generate its semigroup, let both semigroups have a gauge group \mathcal{G} , and $\exists c \in \mathfrak{g}^{\mathbb{R}} : [V_1, V_2] = \text{ad}_c$.⁶ Then $E^{V_1}_t \circ E^{V_2}_s$ and $E^{V_2}_s \circ E^{V_1}_t$ differ only by left composition with “Ad” of an element of \mathcal{G} .

From this theorem, remembering 2.4., immediately follows that the family $\text{Ad}_u \circ E^{V_2}_s \circ E^{V_1}_t$ of $*$ -endomorphisms, where $u \in \mathcal{G}$, $t \geq 0$, $s \geq 0$, is a semigroup. In a more abstract sense, this means a principal bundle[4] over $[0, +\infty) \times [0, +\infty)$ with a semigroup structure corresponding to addition of 2-vectors, with the representation by $*$ -endomorphisms and with the left action⁷ of \mathcal{G} . It is also possible to prove the statement which generalizes the second statement of the theorem of 2.4., but we’ll made such generalisation in Section 4 for a more general case.

Proof: Aforementioned “Ad” term can be obtained by solving a differential equation (by s) in \mathcal{G} , using Lemma 2.6. and the fact that V_2 preserves the gauge algebra. \square

⁵ More strictly, given $\mathcal{A}_{V_1 V_2}$ might have to be extended to some broader $*$ -algebra, such that both $V_1 \circ V_2$ and $V_2 \circ V_1$ are defined and the (new) algebra is preserved by E^{V_1} .

⁶ Note that “ad” of any element of \mathcal{A} is bounded.

⁷ The standard definition of a principal bundle requires a right action, but for this construct, a left action is better suited.

3 Constructions of vector fields

This section will present possible constructions of vector fields via representations of the algebra. Although in simple cases these fields generate semigroups, the section does not contain theorems about it.

3.1. Assume that \mathcal{A} has a representation on $L^2(\mathbb{R})$ (with variable $x \in \mathbb{R}$). The unbounded skew-Hermitian operator $d = \frac{d}{dx}$ of differentiation by x ⁸ is defined on \mathcal{C}^1 -smooth functions, which are dense in L^2 . Suppose that in some dense $*$ -subalgebra the operator $V(q) = d^*q + qd = [q, d]$ is defined. It means that aforementioned operator on $L^2(\mathbb{R})$ is bounded and belongs to the representation. For example, such $*$ -subalgebra would be $\mathcal{C}_0^1(\mathbb{R})$ for $\mathcal{A} = \mathcal{C}_0(\mathbb{R})$. From the product rule for V we see that it is a vector field. If \mathcal{A} is translation-invariant, then V generates a group of $*$ -endomorphisms, called “shifts”.

3.2. The “translation” vector field $V(q) = d^*q + qd$ may be expressed in a matrix form:

$$J := \begin{pmatrix} \mathbf{1} \\ d \end{pmatrix}; \quad J^* = (\mathbf{1} \quad d^*); \quad V(q) = J^* \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} J$$

Or, more complicated:

$$S := \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}; \quad V(q) = J^* S \operatorname{diag}(q, q) J$$

From skew-Hermitianness of “ d ” we have $J^* S J = 0$.

Now let us think that instead of $\operatorname{diag}(q, q)$ we have another representation π of \mathcal{A} , in another Hilbert space \mathcal{D} , which commutes with certain operator S . Suppose that $J : \mathcal{H} \rightarrow \mathcal{D}$ (*transition operator*) is such unbounded operator with dense domain that the operator product

$$V(q) = J^* S \pi(q) J$$

for any $q \in \mathcal{A}_V$ belongs to our (default) representation of \mathcal{A} in \mathcal{H} .

This bears some resemblance to Alain Connes’[2] definition of differentiation, by commutator with the operator F , although we, contrary, try to avoid commutators.

It is easy to check that conditions

$$J^* S J = 0; \quad \forall u, v \in \mathcal{A}_V : [u, J^*] S [J, v] = 0$$

are sufficient to make V a vector field. Note that $[u, J^*]$ means $u \circ J^* - J^* \circ \pi(u)$, an operator from \mathcal{D} to \mathcal{H} , and similarly $[J, v] : \mathcal{H} \rightarrow \mathcal{D}$. Also, if S is self-conjugated, then V is real.

3.3. What appears after replacement of one-dimensional “ d ” with something like Dirac operator? Suppose that J depends linearly on the parameter from \mathbb{C}^2 . Say, there are two operators J_0, J_1 and we define $J_\phi := \phi^L J_L$. Then,

$$V_{\phi \otimes \phi^*}(q) = J_\phi^* S \pi(q) J_\phi; \quad \phi \in \mathbb{C}^2$$

⁸ In quantum-mechanical parlance, $d = i\hat{p}/\hbar$.

The use of tensor product symbol is motivated by an observation that the vector field depends on parameters bilinearly:

$$V_{LL'}(q) = J^*_{L'} S \pi(q) J_L, \text{ or,}$$

$$V_X(q) = X^{LL'} J^*_{L'} S \pi(q) J_L \text{ where } X \text{ is a } 2 \times 2 \text{ matrix.}$$

We can get a 4-dimensional family of vector fields at the price of only 2-dimensional, because J and J^* are essentially the same thing. Though, (for self-conjugated S) only fields parametrized by Hermitian matrices will be real. Note that $\phi \otimes \phi^*$ defines a 2×2 Hermitian matrix with rank not greater than 1 and non-negative trace. We postulated the dependence of J on 2 complex numbers, and got a vector field depending on a 2×2 matrix, which gave the structure of (1+3)-dimensional Minkowski space known in Special Relativity. This construction of the family of vector fields is speculative, but it leads to interesting situation if we suppose that, in some theory, we got that namely $V_{\phi \otimes \phi^*} = J_{\phi}^* S \pi(q) J_{\phi}$ are semigroup-generating, but not other fields in this 4-dimensional complex-linear family.

4 Endomorphism semigroup parametrized by a cone

4.1. The construction from the previous section gives the \mathbb{C}^2 of vector fields of the type $\phi \otimes \phi^*$, which is actually the cone over a Riemann sphere, due to independence of the phase of ϕ . Suppose that all of those fields generate a semigroup.⁹ Let us also suppose that those fields and semigroups has \mathcal{G} as a gauge group and commutators of those fields lie in the gauge algebra.

Obviously, this construction corresponds to the geometry of the cone of the past in the Minkowski space, where 4-coordinates and 2×2 matrices correspond as:

$$X = \begin{pmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{pmatrix} = x^0 I + \sum_{k=1,2,3} x^k \sigma_k, \text{ where } \sigma_k \text{ are Pauli matrices.}$$

Here, the semigroup of translations cannot be generated by two 1-parameter semigroups, so the result from 2.7. is not applicable. BTW, it cannot be generated by *any* finite number of 1-parameter semigroups.

How can we generalize 2.7. for this geometry?

4.2. The spinor representation possibly gave some insight about origins of semigroup-generating vector fields parametrized by the light cone (i.e. by null directions) in the Minkowski space, but it is useless when we have to prove something about semigroups. From here onwards we'll forget anything about

⁹ This is the case, for example, if we defined translations in Minkowski space and our representation is defined on functions on the cone of the past, or another domain containing the whole cone of the past of each of its points. A shift semigroup acts on a domain by filling "overflowed" space with zeros (not explained in this version of the paper).

spinors and think we have n fields V_k and some cone Φ in \mathbb{R}^n such that all fields $x^k V_k$, $x \in \Phi$ generate a semigroup. For the example from 4.1. it would be

$$\Phi = \{ x \in \mathbb{R}^4 \mid x^0 \geq 0, (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 0 \}$$

Let all these semigroups have the same gauge group \mathcal{G} .¹⁰ The condition about commutators takes the form

$$[V_k, V_l] = \text{ad}_{c_{kl}}, \quad c_{kl} \in \mathfrak{g}$$

due to bilinearity of the commutator. Then:

- There exists a principal bundle $F \rightarrow \hat{\Phi}$ (where $\hat{\Phi}$ is a convex hull) with the structure group \mathcal{G} and with a semigroup structure on it which corresponds to vector addition in $\hat{\Phi}$.
- There exists its representation \mathcal{E} by $*$ -endomorphisms of \mathcal{A} which include all semigroups $E^{x^k V_k}$, $x \in \Phi$ such that $E^{x^k V_k}_t$ is parametrized by the point of F corresponding to the point tx^k of Φ .
- The fiber over the 0 point is the group \mathcal{G} itself with its group structure matching the semigroup structure of the bundle. So, endomorphisms from the same fiber differ only by left composition with an Ad_u .
- All (other) vectors $x \in \hat{\Phi}$ have their semigroups $E^{x^k V_k}$ included to F by the same way.

Proof: By applying the theorem of 2.7. to two-dimensional subspaces of \mathbb{R}^n generated by pairs of vectors from Φ we can construct the bundle over such subspaces. Where subspaces intersect, consistency is provided by existence and uniqueness of the corresponding real-parametrized endomorphism semigroup. Iterating this construction $n - 1$ times (or directly applying 2.7.-like reasonings to n -tuples of vectors from Φ) we extend it to all $\hat{\Phi}$.

The vector fields V_k (or, in other words, the field linearly parametrized by an n -dimensional real space) actually define the connection on our bundle, where c_{kl} is its 2-form of curvature. The last (fourth) statement is a manifestation of the fact that such connection can be integrated along the path. \square

References

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- [3] http://en.wikipedia.org/wiki/Ordered_exponential
- [4] http://en.wikipedia.org/wiki/Principal_bundle

¹⁰ This version of the paper lacks an explanation how gauge groups may be related to the spinor construction from Section 3. To be fixed.